Econ 52: Extra Credit Questions*

Spring 2022

The questions in this document are for up to 15% extra credit on Econ 52. Solving any one of these questions is worth 2% extra credit, except where otherwise stated. Solving these questions will substantially improve your understanding of the material covered.

Macro Data

1. **National Income Accounting in a Model:** Can be answered in groups of up to **2**.

   This problem constructs an accounting framework consistent with a simple macroeconomic model to convince you that all the methods of computing GDP should, ideally, indeed give you the same answer. Let time be indexed by t. Start by considering a closed economy with a large number of consumers, indexed by $i = 1, 2, \dots, N$, a large number of firms, indexed by $f = 1, 2, \dots, F$ and a large number of assets, indexed by $j = 1, 2, \dots, J$. In addition to saving in any of these assets, households can save in capital (think of capital being asset number $J + 1$). Assume that there is only one kind of good ("widgets") which are used as both intermediate goods and final goods. The timing of activities in this economy is as follows.

   - Consumer i enters the a period with initial capital k^s_{it}, initial asset holdings a_{ijt} for each asset j and initial shareholdings α_{ift} for each firm f. Shares work as follows[1]: if firm f makes profits π_{ft} then consumer i earns dividend income $\alpha_{ift}\pi_{ft}$ at date t.
   - She chooses how much capital to rent out to firms at the rental rate r_t and how much labor l^s_{it} to supply to firms at the wage rate w_t (the superscript s is there to make it clear that we are referring to labor/capital supply)[2].
   - Production occurs. Firm f decides how much capital k^d_{ft} to rent and how much labor l^d_{ft} to hire. It buys an amount x_{ft} of intermediate goods and using the labor and capital it has, converts these to an amount z_{ft} of widgets, where $z_{ft} \geq x_{ft}$. Firm f's profits are given by

   $$\pi_{ft} = z_{ft} - x_{ft} - w_t l^d_{ft} - r_t k^d_{ft}$$

   - The consumer earns all incomes she is due from labor and capital. She also earns returns r_{jt} on the assets j that she holds, i.e. her asset income from asset j is $r_{jt}a_{ijt}$, and she earns her shares of the profits of all firms in the economy. We use y_{it} to denote her income.
   - The consumer finally chooses how much she wants to consume, c_i, and we define her saving as $s_i = y_i - c_i$. The way consumers save is by purchasing assets that will pay off next period, and investing in new capital. Let a_{ijt+1} be the holdings of asset j that consumer i chooses at date t, which will pay off at date $t + 1$. Normalize the prices of all these assets to 1[3]. Let ι_{it} be gross investment by consumer i in new capital, so that her capital stock next period is

   $$k^s_{it+1} = (1 - \delta)k^s_{it} + \iota_{it}$$

*The questions in this version of this document were written by Aniket Baksy in Spring 2021 - Spring 2022.

[1] We assume here that the only income that comes from shares is dividend income. In reality, a significant amount of the return from holding shares comes from capital gains, which come about when shares are sold at higher prices than they were purchased at. Accounting for capital gains is a tricky business (for instance, the BEA's concept of Personal Income excludes capital gains altogether), and we will not go into this in this problem.

[2] For simplicity, we assume there is a single labor market, in which the price of labor (i.e. wages) is w and there is a single capital market, in which the price of capital is r. Allowing for arbitrary heterogeneity in capital types or labor types or industries and sectors would just add extra summations to all the results we will obtain.

[3] This is just a normalization, since the price of an asset and the returns on the asset have a direct inverse relationship. We normalize terms such that an asset j is a piece of paper that costs \$1 to buy today at date t, and pays the bearer r_{jt+1} next period.

Assume that consumers can hold negative amounts of an asset (this is equivalent to "borrowing" rather than "saving" in an asset), i.e. we allow $a_{ijt+1} < 0$. We make an important assumption: all assets except capital are in zero net supply[4]. That is, we have $\sum_{i=1}^{N} a_{ijt} = 0$ for all assets j and all dates t.

(a) What is firm f's value added? What is Gross Value Added in this economy?

(b) Write down an expression for consumer i's income, y_i, as a function of her factor supplies and share and asset holdings. What is total income earned in the economy?

(c) Show that the value of Gross Value Added you calculated in (a) must equal total income calculated in (b).

(d) Show that total income equals the sum of total consumption and total investment, i.e. that total saving equals total investment.

2. **Living Standards and GDP:**

Go to the World Bank's *Word Development Indicators* Database.

(a) Choose any 5 variables that capture human welfare "better" than raw GDP per capita, in your opinion (or in the opinion of anyone else you've read criticizing the use of GDP). Describe the variables you choose and why you think GDP per capita might not fully capture differences in these variables across countries.

(b) For the years 1990, 2000, 2010, and 2019 (or the latest available year prior to 2020, to avoid the pandemic), plot the values of these variables against GDP per capita. For each year, compute the cross-country correlation between GDP per capita and these variables, and between the log of GDP per capita and the log of these variables wherever the log is meaningfully defined.

(c) Choose any 5 countries for which data coverage is acceptable, and for each of these countries, plot the values of these variables against GDP per capita over time. At least one of your countries should have been a low-income or lower-middle-income country in 1990. For each country, compute the over-time correlation between GDP per capita and these variables, and between the log of GDP per capita and the log of these variables wherever the log is meaningfully defined.

(d) Comment on how strongly GDP and these variables are related to each other, in the cross-section* and in the time-series. What can you infer about the reliability of GDP as a measure of living standards?

NOTE: If you've taken a first course in statistics / econometrics / machine learning, you should use "more advanced" techniques (for instance, regression analysis) or set of statistics to comment on the relationship between GDP and these variables as well.

3. **Understanding Exchange Rates:**

This problem asks you to think about an alternative model for exchange rate determination, based on trade in *assets*. For simplicity, suppose the world consists of two countries, India and the US. Throughout this problem, assume that there are exactly 2 assets - a riskless bond denominated in dollars and a riskless bond denominated in Indian rupees. The annual nominal interest rate in India is i_{IND}%, the annual nominal interest rate in the US is i_{US}%, annual inflation in India is at π_{IND}% and annual inflation in the US is at π_{US}%. Suppose the nominal exchange rate is e_0 INR per USD (INR = Indian Rupee, USD = US Dollar).

[4] An asset is in zero net supply if the creation of the asset implies both the creation of a credit and a debit simultaneously. Consider a debt contract in which Alice borrows from Bob today with a promise to repay tomorrow. As soon as this debt contract is written, Bob acquires a positive position in the asset (Alice's promise to repay tomorrow). But at the same time, Alice acquires an equal and offsetting negative position in the same asset (since she's on the hook to repay the amount tomorrow). Thus, the net increase in saving implied by the creation of this contract is 0. By contrast, when agents save by accumulating capital, there is no offsetting debit created in the economy, and thus capital is in "positive net supply".

(a) Suppose you know the rupee will *depreciate* by $d\%$ between today and a year from now (so that $e_1 = e_0(1+d)$), and that this is certain.

   i. Calculate the returns to saving \$1 in US assets between today and a year from now.

   ii. Calculate the returns to saving \$1 in Indian assets between today and a year from now. Assume that Indian assets are all denominated in INR.

   iii. Say India's nominal interest rate and inflation rate are 7% and 4% respectively, and the US nominal interest rate and inflation rate are 2% and 1% respectively. Suppose d = 1%. Precisely lay out the details of a strategy to make a riskless profit (which assets you will buy and which assets you will sell), and calculate the riskless profit rate per \$ invested in this strategy.

(b) Argue that if international investors have extremely deep pockets, there is a simple formula connecting the gross depreciation rate $1+d$, the gross nominal rates in the two currencies, $1+i_{US}$ and $1+i_{IND}$, and the gross inflation rates, $1+\pi_{IND}$ and $1+\pi_{US}$.

(c) Evaluate the ability of the type of strategy you described in part a,iii to make a riskless profit. Describe intuitively why such a strategy will not make a profit when the relation you developed in part (b) holds, and why it did with the numbers in part (iii) of (a).

Production Functions

4. **A Theory of Cobb-Douglas Production Functions:**

   Where does the Cobb-Douglas Production function come from? One idea is that the Cobb-Douglas is a stand-in for a richer underlying economy, and here's an example. Suppose that the "true" production function for a final goods producer is

$$Y = \exp\left[\int_0^1 \log y(i) di\right]$$

   where each $y(i)$ represents an input of "tasks" that must be performed to produce output. Each task is produced by a competitive intermediate goods firm (this can be thought of as a "department" of the final goods producer), using either by labor or by capital. We assume that capital and labor are perfect substitutes at performing tasks $i \leq \alpha$, but a fraction $1-\alpha$ of tasks cannot be performed by capital at all. That is,

$$y(i) = \begin{cases} k(i) + \ell(i) & i \leq \alpha \\ \ell(i) & i > \alpha \end{cases}$$

   Assume that the price of final output is 1, the prices of labor and capital are $w > 0, r > 0$ respectively. Also assume that $w/r > 1$.

   (a) Suppose each task has a price $p(i)$. Setup the final goods producer's problem, and solve for the demand for each task from the final good producer at these prices.

   (b) Write down the cost function associated with the production of 1 unit of any given task.

   (c) For a given set of prices w, r show that there is a cutoff task $\alpha \in [0,1]$ such that all tasks $i \leq \alpha$ are produced using only capital and all tasks $i > \alpha$ are produced using labor. Calculate the marginal cost of production for each task.

   (d) Note that for the final goods producer, the prices for each task input also represent the marginal costs associated with the production of each task. Use this to substitute for the level of demand for each task, and substitute this level of demand back into the final goods producer's problem.

   (e) Let K, N be the total amount of labor hired by the firm. Show that final output satisfies

$$Y = F(K,N) \propto K^{\alpha} N^{1-\alpha}$$

Factor Markets

5. **Financial Stress and Labor Supply:** Based on Lian, Gorodnichenko and Sergeyev (2022). It's been documented that being poor has significant cognitive costs, which reduces the efficiency units of labor an individual can supply (for instance, because they are distracted by worries about their financial state, or actively engaged in job searches while on the job). Let's study a simple model of this. A household solves the problem

$$\max_{C,N} \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \frac{(N+\theta(Y))^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$$
$$\text{subject to}$$
$$Y = wN + T$$
$$C = Y$$

where T should be interpreted as a combination of non-labor income and transfers net of lump-sum taxes paid. The household takes w, T as given. The function $\Theta(Y)$ parametrizes the disutility that comes from financial stress - the idea is that higher stress raises the disutility of supplying a given number of hours. The specific functional form you can use for $\Theta(Y)$ is $\Theta(Y) = \exp(-Y)$.

   (a) Why is it intuitive that $\Theta(Y)$ should decrease in Y?

   (b) Take first order conditions and eliminate any Lagrange Multipliers that may have shown up.

   (c) Assume for a moment that there is no financial stress (so the function $\Theta(Y) = 0$ for all values of Y). Solve the system of equations you obtained in the previous part on a computer for different values of w. You can use the parameter values $\sigma = 0.5, \theta = 1, T = 0.3$ and values of w in the interval $[0.5, 10]$. Your answer will be a set of values of C, N corresponding to different values of w.

   (d) Now add financial stress back. Using the same parameter values as above, solve for C, N for different values of w. How do the consumption and labor supply functions you computed differ in the stress vs no-stress cases? What kind of data would you want to see to validate this model?

   (e) Now fix w and raise the component T of income. In which case - stress or no-stress - is the response of consumption and labor supply larger? Describe why this is happening. Hint: think of how exactly income and substitution effects work in this case.

6. **Search:** Can be answered in groups of up to **2**.

   This questions constructs what is probably the simplest possible search model, based on McCall (1970). This model is also a gentle introduction to Dynamic Programming, a method for solving dynamic problems that constitutes the bread and butter of modern macroeconomics. An unemployed worker at date t faces a large number of open vacancies, each offering a wage w. Suppose that the wage rate offered is uniformly distributed on $[0, 1]$. At the start of a period, the unemployed worker observes the wage and decides whether or not to accept the wage offer.

   - If she accepts the wage offer, assume that she stays employed at that wage forever, starting today.
   - If she does not accept the wage offer, she must remain unemployed today, earning an unemployment benefit $b < 1$, and then she can search again tomorrow.

   We assume that workers cannot borrow or save, so an unemployed worker's consumption is equal to the unemployment benefit b and an employed worker's consumption is the wage that she accepted, w. Let $\beta < 1$ be the discount factor with which the unemployed worker discounts the future. The worker's utility from a stream of consumption $c_0, c_1, c_2, \ldots$ is given by

$$\mathcal{U}_t = \sum_{\tau=t}^{\infty} \beta^{\tau-t} c_\tau$$

In what follows, it will be useful to note that

$$\mathcal{U}_t = \sum_{\tau=t}^{\infty} \beta^{\tau-t} c_\tau = c_t + \beta \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} c_\tau = c_t + \beta \mathcal{U}_{t+1}$$

so utility can be split into the contribution of consumption today plus the utility the household will earn tomorrow, discounted by the discount factor.

(a) Argue that the utility the worker gets by accepting the wage offer w today is

$$W(w) = \frac{w}{1-\beta}$$

(b) Let U denote the utility the worker gets from waiting one period. Let $\underline{w} = (1-\beta)U$. Argue that the worker must accept all offers such that $w \geq \underline{w}$ and reject all other offers. You will need to assume that $\underline{w} < 1$; for bonus points, describe what happens if this assumption is not satisfied.

(c) Let us try to understand the unemployed worker's decisions more clearly. Start by defining

$$V(w) = \max\{U, W(w)\}$$

Note that $V(w)$ is the utility the worker expects to earn *after* she has observed the current wage rate.

i. Suppose the worker decides to wait one period. Today, she has consumption b. Starting tomorrow, she will again draw a new wage rate, and face the decision of whether to work or not again. Argue that this implies that

$$U = b + \beta \mathbb{E}\left(V(w)\right)$$

ii. If the worker behaves optimally, i.e. as in part (b), argue that the expected utility to the worker tomorrow is given by

$$\mathbb{E}\left(V(w)\right) = \Pr\left(w \leq \underline{w}\right) U + \mathbb{E}\left(W(w) \mid w \geq \underline{w}\right) = \underline{w} U + \frac{1}{1-\beta} \int_{\underline{w}}^{1} w dw$$

iii. By explicitly calculating the value of the integral and using your answer to (a), or any other way, show that

$$\underline{w} = \frac{1 - \sqrt{(1-\beta)(1+\beta(1-2b))}}{\beta}$$

and argue that the other root violates an assumption you made in part (b).

iv. Calculate the job finding rate, which is the probability that the unemployed worker gets a job offer that she accepts. How does this rate depend on b, the value of unemployment benefits, and on β, the discount rate? Provide intuition.

v. Now suppose that with exogenous probability σ, an employed worker gets fired and becomes unemployed next period. Thus, we have

$$W(w) = w + \beta\sigma U + \beta(1-\sigma)W(w) \implies W(w) = \frac{w + \beta\sigma U}{1-\beta(1-\sigma)}$$

Argue that the results of part (a) and (b) and the equation

$$U = b + \beta \mathbb{E}\left(V(w)\right)$$

still hold. Then show that

$$\mathbb{E}\left(V(w)\right) = \Pr\left(w \leq \underline{w}\right) U + \mathbb{E}\left(W(w) \mid w \geq \underline{w}\right) = \left(\underline{w} + \frac{(1-\underline{w})\beta\sigma}{1-\beta(1-\sigma)}\right) U + \frac{1}{1-\beta(1-\sigma)} \int_{\underline{w}}^{1} w dw$$

By explicitly calculating the integral and using part (a), show that

$$\underline{w} = \frac{1-\beta\sigma\tilde{\beta} - \sqrt{\left(1-\beta\sigma\tilde{\beta}\right)^2 - 4\beta(1-\beta)\left(b+\tilde{\beta}/2\right)\left(1-\left(\sigma+\frac{1-\beta}{2\beta}\right)\tilde{\beta}\right)}}{2\beta\left(1-\left(\sigma+\frac{1-\beta}{2\beta}\right)\tilde{\beta}\right)}$$

where $\tilde{\beta} = \frac{\beta}{1-\beta(1-\sigma)}$. Plot $\underline{w}$ against σ for small values of σ. How does the probability that an unemployed worker finds a job depend on σ?

7. **Arbitrage in the Consumer's problem.**

Suppose consumers live for two periods, 1 and 2, and between these dates, they can save in two assets, capital and bonds. For simplicity, suppose the consumer's labor income is exogenously given by W_1, W_2 at the two dates. A unit of capital pays a rental rate r_t^k while a bond pays an interest rate of r_t. The Consumer's problem is

$$\max_{C_1,C_2,B_2,B_3,K_2,K_3} \frac{C_1^{1-1/\sigma}}{1-1/\sigma} + \frac{1}{1+\rho}\left(\frac{C_2^{1-1/\sigma}}{1-1/\sigma}\right)$$

subject to the budget constraints

$$C_1 + K_2 + B_2 = W_1 + (1+r_1)B_1 + (1+r_1^k)K_1$$
$$C_2 + K_3 + B_3 = W_2 + (1+r_2)B_2 + (1+r_2^k)K_2$$
$$K_1, B_1 > 0 \text{ given}$$

(a) Before doing any math at all, argue that $K_3 = 0, B_3 = 0$ and rewrite the problem appropriately. This eliminates two of your decision variables.

(b) Set up the Lagrangian for this problem. Find the first order conditions. Eliminate the Lagrange Multipliers on the two constraints from the system of first order conditions to obtain two Euler Equations.

(c) Compare the Euler Equations and show that $r_2 = r_2^k$. Provide intuition.

8. **Depreciation Rates over time.**

If it feels like they don't make things the way they used to, you aren't alone - owners of the US' nonresidential capital stock would probably agree with you. Annually, the total depreciation of installed nonresidential capital was about 6% in the 1950s and 60s, and this has risen to be over 8%[5]. What are the macroeconomic consequences of this? Let's use the neoclassical model with Cobb-Douglas production to find out. Suppose the depreciation rate goes up suddenly and permanently, with no other changes in the economy. Throughout this problem, focus on the long run.

(a) What happens in the long run to the user cost of capital?

(b) What happens to the target capital stock that firms choose?

(c) What happens to labor demand? Assuming that labor supply is *perfectly inelastic* - that is, the labor supply is fixed at N - what is the impact on the real wage?

(d) What happens to investment?

(e) Look at the data.

  i. **Qualitatively**, are trends in the capital stock and real wages consistent with the predictions of our simple model?

  ii. **Quantitatively**, are trends in the capital stock and real wages consistent with the predictions of our simple model? To do this, start by putting some numbers on various objects. Fix the real interest rate to its long run average, 4%. Find data on the average capital tax rate and the capital share parameter in the production function α. Suppose the depreciation rate was initially 6%, and then rose to 8%. What percent increase (or decrease) do you get in the capital stock and the real wage? What do the data say?

[5]Data from the BEA's Fixed Asset Tables. Download the data at https://apps.bea.gov/iTable/iTable.cfm?ReqID=10&step=2.

9. **Ricardian Equivalence.** Consider an economy that lives for two periods, 1 and 2. There are two agents, a consumer and a government. Consumers can save in government bonds B that they buy from the government. They solve the problem

$$\max_{C_1, C_2, B_2} \log C_1 + \frac{1}{1+\rho} \log C_2$$

subject to the budget constraints

$$C_1 + B_2 = W_1 - T_1$$
$$C_2 = W_2 - T_2 + (1 + r_2) B_2$$

where T_1, T_2 are lump-sum taxes and r_t is the interest earned on government bonds.

(a) Suppose we allowed the consumer to buy bonds at date 2 (i.e. we allowed for a "B_3"). What value of B_3 would the consumer choose?

The Government chooses taxes and borrowings to finance an exogenous stream of consumption G_1, G_2. It faces the budget constraints

$$G_1 = T_1 + B_2, G_2 = T_2 - (1 + r_2) B_2$$

(b) Suppose the government starts with a flat path for taxes, $T_1 = T_2$, with the common level of the tax consistent with the level of government spending. Use the Consumer's first order conditions and the budget constraint to explicitly solve for C_1, C_2 as a function of the exogenous variables W_1, W_2, G_1, G_2, r_2.

(c) Suppose the government wants to push the burden of taxation on to the future as far as possible, and chooses $T_1' = 0$ (and T_2' to be consistent with the budget constraint). What is the impact on C_1, C_2? Are you surprised?

10. **I'm going to the City!:**

The classic Harris and Todaro (1970) model is a cornerstone of development economics, and is one of the most powerful illustrations of the distinction between partial and general equilibrium thinking that economics has produced. This problem asks you to work out the key ideas of the model and apply it to thinking about rural-urban migration in a developing economy context. There is a total mass 1 of people.

The economy consists of two regions, a rural region R and an urban area U. Labor markets in each region are segmented - to work in the rural labor market, you must be physically located in R and to work in the urban labor market, you must be located in U.

All jobs in the rural region are farm jobs, involving working for a landlord who produces using the production function $Y_f = \frac{1}{1-\alpha} N_f^{1-\alpha}$. A landlord's profit maximization problem is just

$$\max_{N_f} \frac{1}{1-\alpha} N_f^{1-\alpha} - w_f N_f$$

By contrast, there are two kinds of jobs in the urban region. High-wage jobs (think of this as being formal sector employment) involve working for a firm that uses a technology that combines capital and labor according to

$$Y = \frac{A_H}{1-\alpha} N_H^{1-\alpha}$$

By contrast, Low-wage jobs (informal sector employment) involves working for a firm that uses no capital, and produces according to $Y = A_L N_L$. Assume that $A_L < 1 < A_H$.

Households are very simple: they maximize their current consumption, and have no disutility of labor supply. They provide one unit of labor in the sector to which they are assigned.

(a) Suppose the distribution of employment across the three sectors is N_f, N_L, N_H in the farm, low-wage and high-wage sectors. What must the wages in each of the sectors be?

(b) Suppose any new arrival in the urban area gets a job in the high-wage sector with probability $N_H/(N_H + N_L)$. What is any migrant's *expected* wage in the city if she migrates?

(c) Suppose that nearly all agents are initially in the rural area. Show that some agents will want to migrate from rural to urban areas.

(d) Migration will proceed until migrants' expected wages in the city just equals their wages in the rural sector. Find a condition connecting N_H, N_L, N_f that makes this happen. Does it matter what the relative values of A_L, A_H are?

(e) Suppose A_H goes up. What is the impact on the three labor markets?

11. **Default Premia in Sovereign Bonds:**

It's the year 1990. Home is Spain, which uses the Peseta, and Foreign is Germany, which uses the Deutschemark. Suppose Nominal interest rates paid to holders of government debt are i_t^S, i_t^D in Spain and Germany (note that nominal returns are always paid in the currency of the issuer). You're a Spanish citizen, earning in Peseta.

(a) What is the return to saving in a Peseta bond?

(b) What is the return to saving in a Deutschemark bond? Note that to buy one of these bonds, you need to first buy Deutschemarks to purchase the bond.

(c) Go online and find some data on interest rates and exchange rates actually prevailing in the two countries in the 1990s. Which of the two strategies would you have pursued? If your answer involves expectations, you can assume that the expected value of a variable at time $t+1$ as of time t is the realized value at time $t+1$ (this is the assumption of *perfect foresight*.

(d) Suppose that there's a large number of international investors with extremely deep pockets, and derive an equilibrium relationship between the two interest rates.

(e) Now suppose that there is a probability δ that Spanish bonds *default*. That is, with probability δ the bondholder is paid exactly 0 and with probability $1-\delta$ she receives $1+i_t^S$. Assuming that investors are risk neutral, re-derive the equilibrium relationship above.

(f) Using the data you collected to answer (c), calculate a time series for the implied default risk δ implicit in Spanish bonds in each year between 1990 and 1999.

(g) Now suppose Spain and Germany choose to adopt a common currency, the Euro. Re-do parts (c)-(f), for the period 2000-2019. How likely did investors think a default was in Spain in the early 2000s?

12. **Capital Flows, Misallocation and Interest Rates:**

This question is designed to get you thinking about what happens in an economy when the interest rate falls for exogenous reasons. Consider an economy with a continuum of agents, each with productivity $A(i) = i$ for $i \in [0,1]$, and a competitive financial sector consisting of banks. Each agent enters the period with one unit of capital. They can choose to

- Save at a bank, which pays them an interest rate of r_t.
- Borrow from the bank, which charges them the same interest rate r_t.

An agent who borrows b has $1+b$ units of capital to produce with. An agent who saves s has $1-s$ units of capital to produce with. The agent produces $y(i) = A(i)k(i)^\alpha$ where $k(i) = 1 + b(i)$ if they choose to borrow and $k(i) = 1 - s$ if they choose to save instead. Assume no capital depreciation and a constant relative price of capital, as well as no taxes.

(a) *The Closed Economy*: Find the interest rate that clears the market. To do this,

i. Argue that r_t is the cost of capital in this economy for *both savers and borrowers.*

ii. Solve for each agent's optimal choice of capital, given r_t. Which agents will borrow and which will save? How much will each borrow/save?

iii. Given the uniform distribution of productivity, find the total amount borrowed and the total amount saved at each interest rate. On a single graph, plot the amount borrowed at each interest rate and the amount saved at each interest rate. You will need to solve an integral to do this analytically. Alternatively, you can use a computer with 1000 agents with productivity uniformly distributed on [0,1] to numerically answer the question.

iv. In equilibrium, the two must be equal. Solve for the interest rate that achieves this. You may need to use a computer to solve for the interest rate.

v. Note that the total supply of capital is 1 since there's a measure 1 of agents who each enter with 1 unit of capital. Use this to find the equilibrium interest rate.

vi. Calculate the output-weighted average productivity of firms.

(b) *The Open Economy*: Suppose you're *given* the interest rate, and the interest rate given to you is lower.

i. Use the same results you had for part (iii) above to find the quantity of borrowing and saving at the new interest rate. Which one is greater?

ii. Calculate the output-weighted average productivity of firms. Provide intuition for whether it is higher or lower.

13. **Sovereign Default.**

Governments often default in situations where repaying the debt is completely feasible, even without levying extra taxes or transfers. One possible explanation for this is that the benefits of being able to issue government debt (stabilization policy, tax smoothing and government investment) net of the costs of repaying the debt are small relative to the benefits of just defaulting, accepting the costs of default and enjoying the extra cash the government has left over since it didn't repay. This question walks you through a simple model to study this tradeoff.

Consider a two-period economy $t = 1, 2$. The economy is populated by a single agent ("the government")[6] that chooses how much to consume at each date c_1, c_2. The economy is endowed with an amount y of goods at each date. The government can borrow in financial markets. That is, it can sell b bonds, each with face value 1 and price q, to raise revenue qb today and allow $c_1 = y + qb > y$ today.

At date 2, the government is on the hook for a repayment b. Note that the (gross) interest rate on government debt is therefore $1/q$. The output in the economy at date 2, y_2, is a random variable that is uniformly distributed between $[0, 2y]$. The outcomes in the economy at date 2 now depend on the government's decision on whether or not to default at date 2.

- If the government chooses to **repay**, it enjoys consumption $c_2 = y_2 - b$ today.
- If the government chooses to **default**, the economy suffers disruption, which we model by assuming that in case of default, output falls to λy_2 where $\lambda < 1$. The government enjoys consumption $c_2 = \lambda y_2$ at date 2.

The government ranks paths of consumption c_1, c_2 by the utility function $U(c_1, c_2) = 2u(c_1) + \beta\mathbb{E}[2u(c_2)]$ where $\beta < 1$, u is an increasing and concave felicity function, and $\mathbb{E}$ is the mathematical expectation operator. We also assume that $y > 1/\beta$.

(a) Suppose we're at the start of date 2. When should the government default? Calculate the probability that the government will default as a function of b and show that there are states of the world where the government could repay the debt (i.e. $y_2 > b$) but chooses not to.

[6] It turns out that in any model where the government is "benevolent" - that is, it has the same preferences as a representative household - and has access to a tax system that lets it control after-tax income precisely, this is without loss of generality. We can just treat the government as making choices $\hat{c}_1, \hat{c}_2$ on behalf of the representative household. The government can implement this chosen path of consumption by choosing taxes in such a way that the household, given its preferences and the path of after-tax income, would optimally indeed choose $\hat{c}_1, \hat{c}_2$.

(b) A government bond is an asset that costs a price q today and pays off an amount 1 in the next period if the government repays, and 0 if the government does not repay. Assuming that bondholders are risk neutral, calculate the price of a government bond, recalling the bond pricing formulas we studied in week 1 and assuming that the interest rate bondholders could have earned elsewhere is r^*. For later, we will assume that $\beta(1+r^*) < 1$.

(c) Suppose $u(c) = c$ and $\lambda = 1/2$ (this is an enormous value for λ!) The government's problem is

$$\max_b y + qb + \beta \frac{1}{2y} \int_0^{2y} \max\{y_2 - b, y_2/2\}\, dy_2$$

Assume that $b < y$ (this assumption must be true in equilibrium, since any value of $b \geq y$ leads to exactly 0 revenue for the government anyway (show this!)). Solve the integral by dividing the domain of integration into two parts, $[0, 2b]$ and $[2b, 2y]$. Solve the resulting problem for the optimal choice of b.

14. **The Small Open Economy and US current account deficits** The US has run an enormous current account deficit for a long time now. Here's a simple model to illustrate why this might be the case. Suppose the world lasts 3 periods, 0, 1 and 2. The world is populated by two countries, the US and the Rest of the World (ROW). Firms in each country c operate the technology $y_c = \frac{A_c}{\alpha} K_c^\alpha$ where $0 < \alpha < 1$. Households in each country have the preferences

$$u(C_{c0}, C_{c1}, C_{c2}) = \log C_{c0} + \beta \log C_{c1} + \beta^2 \log C_{c2}$$

Consumers in the US and in the rest of the world begin date 0 with equal endowments of capital K_0 and no initial debt. Between date 0 and date 1, and between date 1 and date 2, they can borrow or save in capital. Let I_{ct} be the amount invested at date t. Let D_{ct+1} be the amount the household lends abroad (when negative, this is the amount that the household owes to foreigners) at date t (note that the time subscripts are off by 1 - this is a common convention in macroeconomics, and the time subscript is associated with the date at which loans are repaid/debt is due). Let r_t denote the world interest rate (at which US/ROW households borrow/lend from/to each other).

(a) For simplicity, we will assume that capital fully depreciates after production. Show that this implies $K_{ct+1} = I_{ct}$.

(b) Argue that at each date t, the household's budget constraint can be written as

$$C_{ct} + I_{ct} + D_{ct+1} = A_{ct} K_{ct}^\alpha + (1 + r_t) D_{ct}$$

You will need to use the idea that production equals income at some point here.

(c) For simplicity, we will integrate the household and the firm together to obtain the following household problem. Given K_0,

$$\max_{C_{ct}, I_{ct}, D_{ct}, K_{ct}} \sum_{t=0}^{2} \beta^t \log C_{ct}$$

subject to

$$C_{ct} + I_{ct} + D_{ct+1} = \frac{A_{ct}}{\alpha} K_{ct}^\alpha + (1 + r_t) D_{ct}$$

$$D_{c3} = 0$$

Solve this problem for any starting K_0 and the specific sequence of productivity shocks $A_{ct} = 1$ at all dates. To do this,

i. Argue intuitively that $K_{c3} = 0$.

ii. Use the result in (a) to replace I_{ct} at each date and eliminate it as a choice.

iii. Take first order conditions with respect to each choice variable remaining. Show that for $t = 0, 1$,

$$1 + r_{t+1} = A_{ct+1} K_{ct+1}^{\alpha-1}$$

You can assume that $r_0 = 0$ - although this doesn't matter for the problem at all since $D_{c0} = 0$.

iv. Use this equation appropriately to collapse all the budget constraints into an intertemporal budget constraint. What is the household's PVLR?

v. Use the intertemporal budget constraint and the first order conditions linking to calculate consumption at each date as a function of consumption.

(d) Suppose the US and the ROW are mirror copies of each other. Find the unique interest rates r_1, r_2 at which the world economy is in equilibrium: that is, at which it is possible for the amount US households want to lend/borrow to equal the amount ROW households want to borrow/lend. How does this interest rate depend on the A_{ct}'s?

(e) Suppose at date 1, before investment decisions are made, all households learn that the US is about to experience a temporary acceleration in productivity. Specifically, assume that $A_{ROW,t} = 1$ for all $t = 1, 2, 3$ and that $A_{US,0} = 1$, $A_{US,1} = 1 + g$, and $A_{US,2} = 1$. Calculate the interest rates and the amount saved/borrowed at each date.

(f) Identify a notion of the trade balance and the net foreign asset position in this model. What happens to each over time if the shocks to TFP are realized as in the previous part?

Markups, Markdowns and Markets

15. **The Dixit-Stiglitz Demand System**

This problem asks you to work through the Dixit-Stiglitz demand system, which is an incredibly tractable way to model market power in an economy with firms that nonetheless features constant returns to scale production. This is a model of *monopolistic* competition. Consider an economy with N firms and one consumer (or equivalently, a measure 1 of identical consumers).

(a) Suppose that the preferences of the representative consumer are given by the Constant Elasticity of Substitution aggregator. That is, the consumer solves

$$U(c_1, c_2, \ldots, c_N) = \max_{c_1, c_2, \ldots, c_N} \left[\sum c_i^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

subject to the budget constraint

$$p_1 c_1 + p_2 c_2 + \cdots + p_N c_N = PY$$

where $Y = w + \Pi$, and w is total wage income (since there's a measure 1 of households) and Π is the total amount of profits in the economy, which the consumer treats as given. Solve this problem for the demand for each good c_i as a function of p_i, P, Y only, and denote this $c_i(p_i, Y)$. Show that

$$P = \left[\sum_{i=1}^{N} p_i^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

(b) Each good i is made by a different firm with sole rights to produce that Firms operate a constant returns to scale technology with only labor as input. Because firms are monopolists, they choose both their prices and how much they want to produce. That is, each firm $i = 1, 2, \ldots, N$ solves the problem

$$\max_{p_i, y_i, \ell_i} p_i y_i - w \ell_i$$

subject to the constraints

$$y_i = A_i \ell_i \qquad \text{and} \qquad y_i = c_i(p_i, Y)$$

Substitute for the demand curve from part (a) and solve the firm's problem. Show that

$$p_i = \frac{\sigma}{\sigma - 1} \frac{w_i}{A_i}$$

What is the markup, i.e. the ratio between prices and marginal cost for firm i?

(c) General equilibrium requires that in addition to demand equalling supply in all markets $i = 1, 2, \ldots, N$, the labor market must also clear. That is,

$$\sum_{i=1}^{N} \ell_i = 1$$

Use this and previous results to determine the wage rate in equilibrium.

(d) The parameter σ controls the extent to which different goods are substitutable for each other. What happens to markups as the parameter σ falls? Is this intuitive? Hint: what is the price elasticity of demand? Remember how this elasticity determines the extent of market power from Econ 51. Contrast this with what happens when the number of firms N increases.

Economic Growth

16. **Early Growth Models:**

This problem asks you to solve a two pre-Solow/Swan Growth Models: The *Harrod-Domar* Model and the Romer (1986) version of the Frankel (1962) Learning by Doing Model. Consider an economy where a representative firm combines capital K and labor N to produce output. Households can only save by accumulating capital, so savings equal investment. Assume population growth is 0.

(a) The Harrod-Domar Technology assumes that capital and labor are *perfect complements*. That is,

$$Y = F(K, L) = \min\{AK, BL\}$$

Carefully considering the cases $K \leq L$ and $K > L$ separately,

i. What is the marginal product of labor in terms of K, L, A, B?
ii. What is the marginal product of capital?

Assume that the savings rate is exogenous s and that $AK < BL$. Capital accumulates according to the neoclassical equation

$$K_{t+1} = K_t(1 - \delta) + I_t$$

i. Solve for the growth rate of capital.
ii. Can capital grow forever at this rate? Why/Why Not?
iii. Draw paths for capital and output per capita over time.

(b) The Learning by Doing Technology assumes that capital and labor are combined in a Cobb-Douglas fashion to produce final output, that is, firm j's technology is

$$y_{jt} = \overline{A}_t k_{jt}^{\alpha} n_{jt}^{1-\alpha}$$

where $\overline{A}$ is aggregate productivity. However, firms *learn by doing*, so the more capital they install, the more they contribute to the aggregate stock of knowledge. We assume that there are F firms in the economy, a total amount $N = F$ of labor and that

$$\overline{A}_t = \left(\sum_{j=1}^{F} k_{jt} \right)^{\eta}$$

i. Define $Y_t = \sum_{j=1}^{F} y_{jt}$ and analogously define K_t, L_t. Let w_t, R_t be the market-clearing prices of labor and capital respectively (you can think of R_t being the user cost of capital). Show that

A. All firms will choose the same k/n ratio and this will equal K_t/N_t.

B. Show that aggregate output can be written as

$$Y_t = AK_t^{\alpha+\eta}$$

where A depends only on F and is constant over time.

ii. Capital accumulation still follows the same law of motion as above. Show that the growth rate of the capital stock depends on the current level of the capital stock and the quantity $\alpha + \eta$. In particular, show that if $\alpha + \eta > 1$, the growth rate of the economy is itself growing, while if $\alpha + \eta < 1$, then economic growth eventually stops.

17. **Neoclassical Growth:**

This problem asks you to solve a fully specified Neoclassical Growth Model. A representative household inelastically supplies 1 unit of labor and solves the following problem:

$$\max_{\{c_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-1/\sigma}}{1-1/\sigma}$$

subject to the budget constraints

$$C_t + K_{t+1} = (1 + r_t - \delta)K_t + w_t$$

Notice that there is a budget constraint for each date, so your Lagrangian will contain an infinite number of Lagrange multipliers, one for each constraint.

(a) Compute the first order conditions for the household's choice of c_t at each date and derive the Euler Equation.

A representative firm hires capital and labor from the household at rental rate r_t and wage rate w_t. The firm maximizes profits,

$$\max_{K_t, n_t} A_t K_t^{\alpha} n_t^{1-\alpha} - w_t n_t - r_t K_t$$

(b) Compute the firm's first order conditions, one for each factor of production.

Finally, you have market clearing conditions for goods, capital and labor.

(c) Write down the market clearing condition for the capital and labor markets. It turns out that this is sufficient to ensure that goods markets clear too.

Let's characterize a Balanced Growth Path in this economy. Suppose A_t grows at an exogenously given rate g_A. Conjecture that there is a Balanced growth path in which

$$\frac{Y_{t+1}}{Y_t} = \frac{C_{t+1}}{C_t} = \frac{K_{t+1}}{K_t} = 1 + g$$

(d) Show that this requires that the K/Y ratio must be constant.

(e) Show that this requires that the C/Y ratio must be constant.

(f) Solve for the real interest rate in this economy and show that it is constant. Use this to solve for the K/Y ratio.

(g) Solve for the wage rate in the economy, recalling that there is an inelastic supply of labor. Show that it grows at a constant rate.

(h) Solve for the growth rate g as a function of g_A. Use this to solve for the growth rate of the wage rate.

18. **Solving the Two-Period Neoclassical Model on a Computer.** This question is worth up to **10** credit points and can be solved in groups of up to 3 students.

    You can use any programming language you like to do this, but you must be able to define arrays, define functions (in Python, using the "def" command, in MATLAB, creating .m files containing a function), solve nonlinear equations and plot arrays in that language (so, for example, if you're using Excel, you may run into problems.)

    (a) **Set parameters**. This exercise is called calibration of the model, and should help you build intuition for how to put numbers on the parameters we've seen so far. The parameters we need to calibrate are $\sigma, \theta, \rho, \alpha, \delta$. The values of exogenous variables we need to calibrate are A_1, A_2, K_0. The idea of calibration is to pick these parameters and exogenous variable values so that the implications for some endogenous variables are broadly consistent with the data. We will think of one period as one year.

        i. The calibration of θ, ρ is tricky, and so for this exercise, we'll use some standard numbers from the literature. Set $\theta = 1$ and $\rho = 0.05$. These numbers imply that consumers consider utility earned one year from now equivalent to only 95% of utility earned today, and that holding consumption fixed, a 1% marginal increase in the real wage will raise labor supply by 1%. For bonus points, look up some values in the literature for these parameters.

        ii. Collect some data on consumption and real interest rates and find a value of the real interest rate you think is reasonable and a value of consumption growth that you think of as reasonable. Looking through the set of equations, find an equation connecting these objects and calculate the value of σ that is consistent with the calibrated values in (a(i)) and with these data.

        iii. Look up some data on depreciation (this is sometimes called "consumption of fixed capital") and use this to find δ, the fraction of the capital stock that must be replaced each year.

        iv. Recall that with Cobb-Douglas production and competitive markets, the parameter α corresponds to capital's share of income. Use this to choose a value for α that is consistent with the current capital share, which you should look up in the literature. Note that there is limited consensus on how to define the capital stock, capital income, and hence the capital share of income.

        v. Set $K_0 = 1$. This is just a normalization, and it implies that all aggregate quantities Y_t, C_t, K_t we measure should be thought of as multiples of the initial capital stock. Set A_1 to be 1 (again, a normalization) and $A_2 = 1 + g_A$ where you should pick a small number for g_A arbitrarily for now (until part (d)!).

    (b) **Define routines and solve the model.** Construct the function as instructed in the Econ 52 notes document, ch 4, given the values you calculated in (a), and solve for all the endogenous variables. Plot values of output, consumption, investment, hours worked and the capital stock over time.

    (c) **Comparative Statics.** Let's investigate just how the parameter σ affects the endogenous variables of the model. Set up a grid of equally spaced values for σ between 0.5 and 5, the range most macroeconomists would agree is consistent with the data. Solve the model for each value of σ.

        i. In one set of figures, plot the values of output, consumption, investment, hours and the capital stock at date 1 as functions of σ.

        ii. In a separate set of figures, plot the values of the same variables at date 2 as functions of σ (it is good practice to combine the first set of figures into a single plot with multiple subplots, using eg. Matplotlib's subplot command).

    (d) **An internally consistent calibration.** Recall that you picked σ using a value of g_C that you got from the data. Given your preferred calibration, calculate the growth rate of consumption between dates 1 and 2, $C_2/C_1 - 1$. Is this equal to the same value of g_C? If not, the model's calibration is "internally inconsistent." To restore consistency, we need a degree of freedom - and indeed, we have one: the value of g_A, which you picked arbitrarily in (a). By trial and error (or, if you're feeing fancy, using a bisection approach), find the value of g_A such that the model's solution produces the same growth rate of consumption g_C that you used to calibrate σ.

19. **Gerschenkron Effects and Real Growth: Biases in Price Indices**

We learnt in class that $g_{PY} = g_P + g_Y$. The growth rate of real GDP is a critical input into policy formulation, and an important input into the calculation of g_Y is the rate of growth of prices. Unfortunately, some commonly used price indices have fundamental biases built into them. This question asks you to work through a few of these biases.

Suppose the world contains 2 goods, 1 and 2. There is no government spending, the economy is closed, and for simplicity we abstract from capital. Thus, total spending on the two goods must equal GDP. The representative household supplies 1 unit of labor inelastically. Labor is the numeraire - that is, all prices are expressed in units of the real wage rate. The real wage rate is therefore identically equal to 1 (think of changing the units of all prices in the US from Dollars to "22 dollar equivalents."). The two goods are produced using the technologies

$$Y_i = A_i n_i, i = 1, 2$$

Firms therefore maximize profits $\max_{p_i} p_i A_i n_i - n_i$.

Suppose you know that[7] the utility function is $U(c_1, c_2) = \log c_1 + \log c_2$. Households solve the problem

$$\max_{c_1, c_2} U(c_1, c_2) \qquad \text{subject to } p_1 c_1 + p_2 c_2 \leq (n_1 + n_2) = 1$$

taking w as given.

(a) What are the only values of p_i consistent with equilibrium in labor and capital markets here? Hint: what would happen to labor demand if $p_i > 1/A_i$?

(b) Given p_1, p_2, w, solve for optimal choices of c_1, c_2. Use this to figure out what n_1, n_2 must be.

(c) Calculate a Price Index for this economy, $CPI = p_1 c_1 + p_2 c_2$. This should only be a function of A_1, A_2.

(d) Now suppose parts $(a) - (c)$ were all about describing a first period $t = 0$. In period $t = 1$, A_1 rises to $2A_1$. Redo parts (a)-(c) for period 1, and find the change in the price index.

(e) Calculate *Laspeyre's* Price Index at the two dates, defined as

$$P_t^L = \frac{p_{1,t} c_{1,t=0} + p_{2,t} c_{2,t=0}}{p_{1,t=0} c_{1,t=0} + p_{2,t=0} c_{2,t=0}}$$

Find the change in the Laspeyre's Index, and argue that the use of a Laspeyre's index will lead to an overestimation of inflation (or an underestimation of deflation).

(f) Calculate *Paasche's* Price Index at the two dates, defined as

$$P_t^P = \frac{p_{1,t} c_{1,t=1} + p_{2,t} c_{2,t=1}}{p_{1,t=0} c_{1,t=1} + p_{2,t=0} c_{2,t=1}}$$

Find the change in the Paasche's Index, and argue that the use of a Paasche's index will lead to an underestimation of inflation (or an overestimation of deflation).

20. **"Schumpeterian" Growth**. This problem asks you to work through a simple growth model inspired by the ideas of Schumpeter. The economy has J industries indexed by $j = 1, 2, 3, \ldots, J$. At any date in this industry, there is a leader firm l who can produce output using the technology

$$Y_{ijt} = A_{it} n_{it}$$

The representative household has preferences over the goods given by

$$U(c_{1t}, c_{2t}, \ldots, c_{jt}) = \sum_{j=1}^{J} \left(c_{jt}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}$$

[7]The logic of this question works for any concave utility function.

For simplicity, there is no borrowing or lending. The households simply consume all their income each period, and they derive income solely by working a total of N hours across all sectors. The budget constraint of the household is

$$p_{1t}c_{1t} + p_{2t}c_{2t} + \cdots + p_{Jt}c_{Jt} = \underbrace{\sum_{j=1}^{J} w_t n_{jt}}_{W_t} = w_t N$$

(a) Calculate the demand curve for each industry's good, which gives you the total demand c_{jt} as a function of W_t, p_{jt} and parameters.

(b) The leader firm of industry j is free to choose any price and quantity to produce, but is subject to the constraint imposed by the demand curve. Find the profit maximizing choice of p and y. Using this, calculate the profits made by the monopolist. Call these profits $\pi(\cdot)$, where $\pi(\cdot)$ is a function of the wage rate/ wage income, parameters and, importantly, of A.

(c) At the end of each period t, a new firm is selected to be a potential leader next period. The potential leader now makes an innovation investment s. Given this investment s, she succeeds with probability λs^θ where $\lambda > 0, 0 < \theta < 1$ are constants. If she succeeds, the industry's productivity rises to $A_{t+1} = \gamma A_t$ with $\gamma > 1$ she gets the monopoly profits $\pi_{t+1}(A_{t+1})$. If she fails, she gets a value of 0 and the sector's productivity remains $A_{jt+1} = A_{jt}$. The potential leader's problem is to choose investment to maximize her expected value, i.e.

$$\max_s -s + \lambda s^\theta \pi_{t+1}(A_{t+1})$$

What is the optimal choice of s as a function of A?

(d) Suppose all sectors are symmetric. On average, how many of them will grow by a factor γ and how many will not? How fast does the whole economy grow?

The US Economy

21. **"Local" Business Cycles:**

This question can be answered in groups of up to 3, each of whom will receive 2 credit points. On the Bureau of Economic Analysis' website, find information at the county-level on GDP and income per capita at an annual frequency.

(a) Using any statistical method you like, document whether income across counties moves in tandem with aggregate income in the US. That is, when the US is in a boom, do most counties on average also experience booms? Are there any counties whose incomes are countercyclical? **Note:** This question is vague on purpose. You are expected to come up with your own approach to answering the questions here. Be creative! Some concepts you may find useful:

- The sample time-series expectation of a variable X_t is defined as

$$\mathbb{E}(X_t) = \frac{1}{T}\sum_{t=1}^{T} X_t$$

- The sample time-series covariance between two variables X_{1t}, X_{2t} is defined as

$$\mathbb{C}(X_{1t}, X_{2t}) = \frac{1}{T-1}\sum_{t=1}^{T} [X_{1t} - \mathbb{E}(X_{1t})]\,[X_{2t} - \mathbb{E}(X_{2t})]$$

- Some readings to consider for ideas: here and here.

(b) Provide interpretations of your findings. To do this, pick a set (at least 5) of counties whose incomes are strongly correlated with aggregate conditions, whose incomes are roughly acyclical, and whose incomes are countercyclical. For each of these counties, describe in detail why you think their incomes covary with aggregates in this specific way.

22. **The US Business Cycle:**

This question asks you to work out some properties of the business cycle and estimate TFP. Doing this yourself is a rite of passage for Macroeconomists, and gives you a bird's eye view of complexities associated with working with aggregate data. **This question is worth up to 3 credit points.**

Choose any country you like. This country must have a relatively reliable data collection service producing statistics on national income accounts at least 40 years. The US is an easy choice, but I strongly encourage choosing a different one. Spain, for instance, has fantastic data.

(a) Find data on GDP, Consumption of Durables, Consumption of Non-Durable goods, Investment, Government Spending, Hours worked, Employment, Wages and Nominal interest Rates. Convert the quantity series to logs.

(b) Detrend the series in any way you like (the preferred way in Macroeconomics is to use a *filter* like the Hodrick-Prescott filter, but you should use any method you like. A cubic time trend is fine for many purposes. Ask a TA for more details on the HP filter.).

(c) Compute the time-series correlations between these detrended variables and detrended log GDP. Compute the standard deviations of the detrended series, and compare these with the standard deviation of detrended log output. Do the business cycle facts documented in the videos hold up?

(d) Compute TFP, assuming that production is Cobb-Douglas. You will need to decide whether to use contemporaneous measures of the capital stock or lagged measures, and you'll need to decide how to choose the hours measure (the US, for instance, has a number of options for you in terms of choosing how to measure hours worked.). You'll need a way to calibrate α, the labor share parameter, as well. Your grade on this question will depend on how well you justify the choices you make.

23. **The Empirics of Long-Run Structural Change in the US:**

This question is for those of you with an empirical bent of mind and some aptitude in working with big datasets. **This question can be answered in groups of up to 3 people, all of whom will receive up to 4% credit.**

(a) Go to the website for IPUMS USA, where you can download public-use microdata from the US Census. Download data for the full-count censuses from 1870-1940 (warning: the files are LARGE.).

(b) The data include information on basic demographics (race and sex), as well as on the industrial sector and the occupation worked in. Use this data to answer the following questions. Be sure to clearly document any data cleaning steps you use.

i. What was the industrial composition of the US like in the late 19th century? What were the main industries by employment? How did the dominant sectors of the US economy change over this period? Read up on US economic history to understand the trends you find in more detail.

ii. Statisticians and Economists have devised measures of segregation, i.e. statistics that can be used to quantify the extent of segregation between two groups of individuals. Go online and find out about these statistics. Can you find evidence of occupational segregation in the US by race over this period, and does segregation shrink over time? Is there a geographic dimension to this?

Money and the Economy

24. **The Quantity Theory of Money: A Micro-foundation.** Consider a simple economy. A firm produces output with only one factor of production, labor, according to the production function $Y_t = Z_t N_t$ where Z_t is a measure of labor productivity and N_t is labor input. Firms maximize profits $P_t Y_t - W_t N_t$ each period, where P_t, W_t are the price level and the nominal wage rate respectively.

    On the household side, H identical households solve the problem

    $$\max_{C_t, N_t} \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \frac{N_t^2}{2} \right)$$

    subject to the budget constraints

    $$P_t C_t + B_{t+1} + M_{t+1} = W_t N_t + (1 + i_t) B_t + M_t$$

    In this problem, households enter the period with cash M_t and bonds B_t. Bonds then mature, and the household starts with cash on hand $M_t + (1 + i_t) B_t$. However, they must hold cash to pay for consumption. That is, if they choose to buy bonds worth B_{t+1} to take into the next period, their consumption C_t satisfies

    $$P_t C_t + B_{t+1} \leq M_t + (1 + i_t) B_t$$

    This constraint is called the "cash in advance" constraint.

    Only after this is over does the household earn wages $w_t N_t$. These wages are paid in cash, so the amount of money the household enters the next period with is $w_t N_t$ plus whatever cash was left over after spending on consumption and bonds for tomorrow. Note that C_t is the real level of consumption (i.e. consumption in terms of goods), and $P_t C_t$ is therefore the value of consumption in units of money (say dollars). B_{t+1} and M_{t+1} are the holdings of bonds and money (in the form of cash) that households have on hand at the end of the day. Given an exogenous path for money supplies M_t^S, an equilibrium is a path for consumption, bond and money holdings by households that maximize the households' utility subject to both constraints, the firms' profits and lead to market clearing in the markets for bonds, money, labor and goods:

    $$\begin{aligned} M_t &= M_t^S > 0 \\ B_t &= B_t^S > 0 \\ Y_t &= C_t \end{aligned}$$

    (a) Suppose $Z_t > W_t / P_t$. What does the firm's demand for labor look like? Suppose that $Z_t < W_t / P_t$. What does the firm's demand for labor look like? What must the real wage be in equilibrium?

    (b) Assume that the cash in advance constraint holds with equality. Write down the Lagrangian for the household problem, with multipliers $\beta^t \lambda_t$ and $\beta^t \nu_t$ on the date-t budget constraint and the date -t cash in advance constraint. Note that the way we just rescale the multipliers by β^t only makes the math a bit prettier. You don't have to do this - just ignore the β^t if you want to.

    (c) Find the first order conditions with respect to C_t, N_t, B_t, M_t.

    (d) Use the first order conditions and the Fisher equation to first show that the Euler equation you've derived throughout this course is still valid. Derive a relationship between i_t, λ_t, ν_t.

    (e) Show that

    $$M_t = P_t Y_t$$

    by combining the first order conditions with the date-t budget constraint and the results derived earlier on the wage rate.

25. **Money in Utility:** This question can be answered in groups of up to 2 students.

    Let's study a simple economy with three assets - Money, Bonds and Productive Capital. One theory for why people hold money is that money is liquid, and hence makes experiencing life easier, which we model by assuming that the household's utility function includes money directly. A representative household inelastically supplies a unit of labor and solves the problem

$$\max_{\{C_t, K_{t+1}, M_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[u(C_t) + v\left(\frac{M_{t-1}}{P_t}\right)\right]$$

$$\text{subject to } C_t + T_t + K_{t+1} + \frac{Q_t B_t}{P_t} + \frac{M_t}{P_t} = w_t + (1 - \delta + x_t)K_t + \frac{B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t}$$

where

- C, K, w are consumption, capital and the real wage rate
- B is the number of bonds purchased today with face value 1, at price Q.
- P is the price level in terms of goods
- T is the amount of lump-sum taxes paid by households
- M is the amount of money held by households in nominal terms

A representative firm rents capital at the real rental rate x_t and solves the problem

$$\max_{K_t, N_t} K_t^\alpha N_t^{1-\alpha} - w_t N_t - x_t K_t$$

The government collects taxes, borrows and prints money. Its budget constraint is

$$G_t - T_t + \frac{Q_t B_t^G}{P_t} = \frac{B_{t-1}^G}{P_t} + \frac{M_t - M_{t-1}}{P_t}$$

We assume that G_t, T_t are exogenous. Define

- $r_t^k = x_t - \delta$
- $Q_t = 1/(1 + i_t)$, which defines the nominal interest rate
- $\pi_{t+1} = \frac{P_{t+1} - P_t}{P_t}$
- $1 + r_t = (1 + i_t)/(1 + \pi_{t+1})$
- $m_t = M_{t-1}/P_t, b_t = B_{t-1}/P_t$.

(a) Carefully write down the market clearing conditions for goods, labor, capital, bonds and money.

(b) Using the household's first-order conditions, show that $r_t = r_t^k$.

(c) Show that $v'(m_{t+1}) = i_t u'(C_{t+1})$, and provide intuition.

(d) Suppose $M_{t+1} = M_t(1 + \mu_t)$. Show using optimality conditions for M, B that

$$m_t = \frac{m_{t+1}}{(1 + \mu_t)(1 + r_t)} \left(1 + \frac{v'(m_{t+1})}{u'(c_{t+1})}\right)$$

(e) Calculate the government's revenue from seignorage.

(f) Rearrange the optimality conditions of the economy to obtain 5 equations in the variables Y, C, K, x, r, noting that G is exogenous. Hence show that none of these variables depends on money - that is money is neutral.

(g) Impose steady state, so that all real aggregates Y, C, K are constant. Normalize all variables so that at the steady state $u'(C) = 1$ (we can always do this by changing units). Show that the only way to have a monetary equilibrium with $\mu > 0$ (i.e. with positive growth in money supply) is if there is a solution to the equation

$$v'(m) = (1 + r)(1 + \mu) - 1$$

(h) If the steady state above exists, show that prices must grow exactly as fast as the growth rate of money supply.

Modern Economic Models

26. **Production Networks.** Let's study the implications of modeling intermediate goods production in the economy. The following is based on Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi (2012). Suppose there are 3 sectors, numbered $i = 1, 2, 3$. We will use the following notation.

    - Sector i's final output will be denoted y_i.
    - The amount of goods produced by sector j and used as an intermediate in production by sector i will be denoted x_{ij}.
    - The amount of goods produced by sector i for final consumption is c_i.
    - The amount of labor used by sector i is ℓ_i.
    - p_i is the price of sector i's output.

    Sector i produces output using the production function

    $$y_i = e^{z_i} \ell_i^{\alpha_i} x_{i1}^{\alpha_{i1}} x_{i2}^{\alpha_{i2}} x_{i3}^{\alpha_{i3}}$$

    A representative household supplies one unit of labor inelastically. The household chooses consumption of all three goods to maximize the utility function

    $$U_t = \log c_1 + \log c_2 + \log c_2$$

    subject to the budget constraint $p_1 c_1 + p_2 c_2 + p_3 c_3 = w$.

    (a) Consider two equations,

    $$y_i = c_i + x_{i1} + x_{i2} + x_{i3}$$
    $$y_i = c_i + x_{1i} + x_{2i} + x_{3i}$$

    Argue that the second of these is the correct one.

    (b) Solve the household's utility maximization problem to find c_i as a function of labor income w.

    (c) Set up and solve each sector's profit maximization problem to find input demands and supplies of output given the price p and the wage rate w.

    (d) Use the results derived above and the labor market clearing condition $\ell_1 + \ell_2 + \ell_3 = 1$ to get a system of 4 equations in 4 variables p_1, p_2, p_3, w. Solve this system. Using matrix algebra is STRONGLY recommended.

    (e) Show that log GDP is proportional to the vector of *gross* sales shares, $p_i y_i / (p_1 y_1 + p_2 y_2 + p_3 y_3)$.

Price Stickiness

27. **Time Inconsistency:** Can be answered in groups of up to 2 students.

    Why do Central Banks need inflation targets? One possibility is that they may be tempted to inflate the economy otherwise, as argued by Kydland, Prescott, Barro, Gordon and a number of others. Let's show this. Consider the following Dynamic Stackelberg game with Uncertainty. Events occur in the following order.

    - The public chooses expected inflation $\pi^e = \mathbb{E}$.
    - A shock ϵ hits the economy, and for simplicity assume that $\epsilon \sim N(0, \sigma^2)$
    - The central bank chooses what inflation will actually be realized (eg by setting money growth to a certain number). This choice is made to minimize the loss function

      $$L = \min_{\pi} \pi^2/2 + \lambda(u - u^T)^2/2$$

      where u^T is a target inflation rate subject to the Phillips curve described in the next point. We assume that $u^T = u^N - k$ where u^N is the natural rate of unemployment.

- Unemployment outcomes are realized, via the Phillips Curve $u = u^N - (\pi - \pi^e) + \varepsilon$.

We'll use a technique called backward induction to solve this game.

(a) Start at the third stage. The Central bank takes π^e and the shock ε as given and chooses π to minimize its loss function, given the Phillips curve and the fact that $u^T = u^N - k$. Solve for the optimal choice of inflation.

(b) Given the optimal choice of inflation, solve for expected inflation by households in the first stage.

(c) What is realized unemployment in the economy? Does the Central Bank hit its target? Explain why/why not.

(d) Call the solution above the solution **with discretion**. Now let's solve for the solution **with commitment**. For this, suppose we force the central bank to choose $\pi = 0$ no matter what happens. Calculate the loss to the Central bank under discretion and under commitment.

(e) Suppose we could *force* the Central bank to always hit a certain inflation rate π^*. Calculate the optimal target rate that the economy would choose. What does your answer depend on, and why?

(f) **Bonus:** Suppose we could define a set of narrow bands for inflation, allowing the central banker to choose any inflation rate within this band. Show that for narrow enough bands that are still non-zero in width, this would make society strictly better off than a fixed inflation target.

28. **Calvo Price Setting: Some Real Math**

This question is worth **10 extra credit points**.

The key innovation of a New Keynesian Model over a Real Business Cycle model is the Price Setting section*, which takes seriously the problems firms may face in adjusting their prices. This problem walks you through a "reduced form" way to model price stickiness - the idea of a "Calvo Fairy". The idea is that as a firm, each period, there's a certain probability that you will be able to change your price. With complementary probability, you're stuck with your current price, and have to adjust how much output you're willing to produce in response to this.

Let's first consider a firm with technology $Y_t(i) = A_t N_t(i)$ (ignore capital). Firms face a demand function $Y_t(i) = (P_t(i)/P_t)^{-\varepsilon} Y_t$ where you can show that ε is the elasticity of demand and Y_t is aggregate demand. Firms can hire any amount of labor they want in a competitive market, with wage rate w_t.

(a) Calculate the nominal total cost C of producing an amount $Y_t(i)$. We will use the notation $C_t(Y_t(i))$ is the total cost function.

(b) Show that the nominal marginal cost function $C_t(Y_t(i)) \equiv C_t$ doesn't depend on $Y_t(i)$ and depends on w_t, A_t, P_t where w_t is the real wage rate, P_t is the aggregate price level and A_t is productivity.

(c) Each period, the firm faces a constant probability θ of being able to change its price. Suppose the firm sets its price today ($t = 0$). What is the probability that the price is still being used l periods from today? What is the fraction of firms who are still using a price set l periods ago?

(d) The firm solves the problem

$$\max_{P_t^*} \mathbb{E}_t \sum_{l=0}^{\infty} \beta^l \theta^l \left(\frac{(P_t^* - c_t) Y_{t+l}(i)}{P_{t+l}} \right) \qquad \text{subject to } Y_{t+l}(i) = \left(\frac{P_t^*}{P_t} \right)^{1-\varepsilon} Y_{t+l}$$

Calculate the first order condition with respect to P_t^*. It is a good idea to first substitute for $Y_{t+l}(i)$ using the constraint.

(e) Now suppose that there is a continuum of firms of measure 1. Note that every firm changing its price at the same time must choose the same new price, irrespective of their current prices. This is a consequence of the price setting decision being purely forward looking. Let P_{t-k}^* be the optimal price set by firms k periods before the current date t. Characterize the entire distribution of prices - that is, the set of prices and the fraction of firms at each of those prices.

(f) It can be shown using a household problem that the appropriate price index is

$$P_t = \left(\int_0^1 P_t(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

Show that

$$P_t = \left[(1-\theta) \sum_{l=0}^{\infty} \theta^l P_{t-l}^{*}{}^{1-\varepsilon} \right]^{1/(1-\varepsilon)}$$

(g) It can be shown that in log-deviations from steady state[8], we get

$$\hat{p}_t^* = \beta\theta \mathbb{E}\hat{p}_{t+1}^* + (1-\beta\theta)(\hat{p}_t + \nu \hat{k}_t)$$

where $\nu = 1/(1+\varepsilon)$ and $\hat{k}$ is the log deviation of real marginal costs from steady state values. It can also be shown that the price level satisfies

$$\hat{p}_t = \theta \hat{p}_{t-1} + (1-\theta)\hat{p}_t^*$$

Combine these equations at t and $t+1$ and eliminate $\hat{p}_t^*$ to get an equation connecting log deviations of inflation $\hat{\pi}_t = \hat{p}_t - \hat{p}_{t-1}$ to expected inflation $\mathbb{E}\hat{\pi}_{t+1} = \mathbb{E}\hat{p}_{t+1} - \hat{p}_t$ and $\hat{x}_t$. This is the (in)famous **New Keynesian Phillips Curve**.

29. **Inequality and the Macroeconomy: Uninsurable income risk.**

This problem asks you to show that if individuals had access to perfect financial markets that allowed them to share risk via insurance contracts, there would never be any inequality in the economy. This question is worth **3** credit points.

The economy contains 2 agents, A and B. Total output in the economy is fixed at 1, but the distribution of this output is unequal. In particular, each period, with probability 1/2, A has income $y_A = 1/2 + \varepsilon$ and B has income $y_B = 1/2 - \varepsilon$. With the complementary probability, A has income $y_A = 1/2 - \varepsilon$ and B has income $y_B = 1/2 + \varepsilon$. A and B cannot borrow or lend. They discount the future at rate $\rho > 0$ and value consume according to the expected present discounted value of lifetime consumption

$$U_i = \mathbb{E} \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t u(c_{it})$$

where $i = A, B$ and $u(\cdot)$ is a concave function (that is, it satisfies $u((x+y)/2) \geq (u(x) + u(y))/2$.

(a) Suppose A and B don't talk to each other. How much does each consume in each possible state of the world?

(b) Calculate each agent's expected present discounted value of lifetime consumption under this assumption.

(c) Now suppose A, B talk to each other and come up with the following arrangement. For all periods hence,

- If $y_A = 1/2 + \varepsilon$ then A will pay B an amount ε
- If $y_A = 1/2 + \varepsilon$ then A will pay B an amount ε

Repeat the previous parts for this case.

(d) Why is it appropriate to think of this arrangement as a kind of insurance contract?

(e) Would agents rather live in the world without the contract or the one with the contract? How much could a hypothetical insurance company charge each of the two parties to make them still willing to accept the contract?

[8]This uses a Technique called log-linearization.

(f) How much inequality is there in the economy with the contract?

30. **Race and the Economy.**

The US has a notoriously high level of racial inequality. Has any progress been made at all in closing the racial welfare gap? To answer this, suppose that welfare for an agent born at date t is given by the present discounted value of lifetime utility

$$\sum_{a=0}^{X} \prod_{s=0}^{a} (1-\delta_s)\,(\beta)^t \log c_{a,t}$$

where $c_{a,t}$ is the consumption of an agent born at date t when she is a years old (i.e. at date $t+a$) and recall the notation $\prod_{s=0}^{a}(1-\delta_s) = (1-\delta_0)(1-\delta_1)\ldots(1-\delta_a)$. X is the maximum lifespan an agent has and δ is the death rate. This calculation will therefore capture two dimensions of inequality across races: in mortality and in consumption. Since it excludes other dimensions, this is something of an underestimate of inequality.

(a) Let's choose two years, $y_0 < y_1$. You can choose any two years with $y_0 \leq 1990$ and $y_1 \geq 2000$. For each of these years, construct the following data.

i. The probability of dying within one year as a function of age and race. Consider only two racial groups, whites and african-americans. This website will be useful. If you obtain data that groups ages into 5 year intervals (ages 0-5, 5-10, ...) then assign the same death rate to each of the ages in a given interval. This gives you $\delta_s, s = 1, 2, \ldots, X$.

ii. Consumption of agents of age a for the two racial groups. This is a bit harder to get. Assume that agents' consumption can be proxied for by their income. Use microdata from the March CPS to construct this, available on David Autor's website here. (the data files are available in Stata format, which you can convert to csv files easily enough using Stata on Farmshare). Note that the data contain an hourly wage rate, which you need to convert to annual income appropriately using other variables available in the dataset. Take an appropriate average over agents to get c_{at}, the consumption of an agent born at date t of age a.

(b) Assume $\beta = 0.95$, a typical value in macroeconomic studies using annual data.

i. Construct the survival probabilities up to age a for each age group, given by $\prod_{s=0}^{a}(1-\delta_s)$.

ii. Compute the PV of utility for an agent of each race at each date. How much has white welfare risen? How much has black welfare risen? By how much would we need to multiply black consumption at all ages to equate white and black welfare? Comment on current debates surrounding reparations using your calculations and benchmarking them to estimates of reparations proposed by prominent figures.